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A high temperature series study of the ANNNI model in two and three dimensions

J Oitmaa

School of Physics, The University of New South Wales, Kensington NSW 2033, Australia

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Abstract. We have derived eleven-term high temperature series for the wavevector dependent susceptibility $\chi(q)$ of the ANNNI model in two and three dimensions. In three dimensions the locations of the phase boundaries of the ferromagnetic and modulated phases, and the location of the Lifshitz point, are found to be in good agreement with previous work. In two dimensions the analysis is less precise, but yields results consistent with the currently accepted picture in which the paramagnetic phase extends to zero temperature at the multiphase point.

1. Introduction

There is currently much interest in physical systems with spatially modulated phases, which can be either commensurate or incommensurate with the underlying lattice. Such behaviour can be found in a variety of systems—for example, certain magnetic compounds, adsorbed monolayers on surfaces, charge density wave systems.

A particularly simple model which exhibits a modulated phase, and which is therefore of interest for understanding such behaviour in real systems, is an Ising model with competing ferromagnetic and antiferromagnetic interactions along one lattice direction—this is the so-called ANNNI model. Much work has been done on this model during the last five years, using a variety of complementary techniques, and its properties are now fairly well understood. We do not attempt here to survey the extensive literature, but refer to a recent review by Selke (1984). A good introduction to the general topic of commensurate and incommensurate phases has been given recently by Bak (1982).

The ANNNI model is shown in figure 1, and consists of a set of planes (or lines) with nearest-neighbour interactions of strength J_0 which are coupled in the z direction by nearest-neighbour interactions of strength J_1 and next-nearest-neighbour interactions of strength J_2. Thus we may write the Hamiltonian as:

$$\mathscr{H} = -J_0 \sum_{\langle ij\rangle}^{(x,y)} \sigma_i\sigma_j - J_1 \sum_{\langle ij\rangle}^{(z)} \sigma_i\sigma_j - J_2 \sum_{\langle ij\rangle}^{(z')} \sigma_i\sigma_j \tag{1}$$

with $\sigma_i = \pm 1$. The zero-field energy of the system is unchanged on changing the signs of J_0 and/or J_1, so that without loss of generality we take these to be positive (ferromagnetic). J_2 is taken to be of either sign, although the most interesting case is when J_2 is negative, corresponding to competing antiferromagnetic second-neighbour

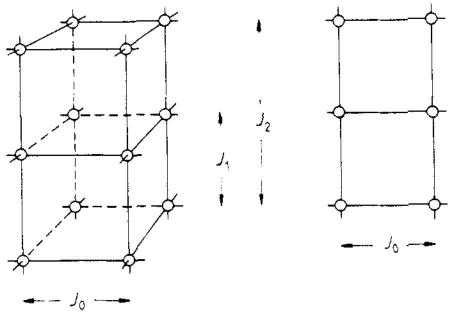


Figure 1. The ANNNI model in three and two dimensions.

interactions in the z direction. It is this competition which gives rise to the possibility of a modulated phase.

For completeness we give a brief discussion of the predictions of mean-field theory for this model. The wavevector dependent susceptibility in the paramagnetic phase, within this approximation, is given by:

$$\chi(q)=\chi_0/[1-\beta J(q)] \tag{2}$$

where χ_0 is the single spin susceptibility, $\beta=1/k_BT$, and

$$J(q)=4J_0+2J_1\cos(2\pi q)+2J_2\cos(4\pi q). \tag{3}$$

In writing the expression (3) we have specialised to the three-dimensional case with wavevector in the z direction. It is easy to see that when $J_1+4J_2>0$ the quantity $J(q)$ has a maximum at $q=0$, and hence as the temperature is lowered the susceptibility diverges first at $q=0$, corresponding to a transition to a ferromagnetically ordered phase. However when $J_1+4J_2<0$ the susceptibility diverges first at a critical wavenumber $2\pi q_c=\cos^{-1}(-J_1/4J_2)$, corresponding to a transition to a phase whose magnetisation is modulated in the z direction with this characteristic wavenumber. The point $J_2=-\frac{1}{4}J_1$, with transition temperature $k_BT_c=4J_0+1.5J_1$, is thus a Lifshitz point (Hornreich *et al* 1979), a point at which disordered, ferromagnetic and modulated phases meet. Bak and von Boehm (1980) have analysed the structure of the modulated phase itself and find a 'devil's staircase' structure of commensurate phases. At $T=0$ the ordered state is ferromagnetic for $J_2>-\frac{1}{2}J_1$ and a commensurate state with $q=\frac{1}{4}$ (consisting of two up layers, followed by two down layers, etc) for $J_2<-\frac{1}{2}J_1$.

Various methods have been used to go beyond mean-field theory. While not attempting to be comprehensive, we mention the high temperature series work of Redner and Stanley (1977a, b), the low temperature series approach of Fisher and Selke (1981), and the Monte Carlo work of Selke and Fisher (1979) and Rasmussen and Knak Jensen (1981) for the three-dimensional case. In two dimensions there are Monte Carlo results (Selke 1981), analytic approximations (Villain and Bak 1981, Kroemer and Pesch 1982), and series work (S Redner, unpublished). In addition there are studies of a one-dimensional quantum analogue (Barber and Duxbury 1982, Duxbury and Barber 1982, Rujan 1981).

In this paper we present the results of a study of both the two- and three-dimensional versions of the ANNNI model. Using a high temperature multigraph expansion formalism (Oitmaa 1981) we have derived and analysed eleven-term series for the wavevector dependent susceptibility $\chi(q)$. This work should thus be regarded as an extension of the work of Redner and Stanley (1977a, b), who obtained an eight-term series using the Wortis linked-cluster method. We discuss, briefly, the method of derivation of the series in the following section. Section 3 is devoted to an analysis of the three-dimensional series, while the analysis for the two-dimensional case is given in § 4. In § 5 we summarise our results.

2. Calculation of the series

The wavevector dependent susceptibility $\chi(q)$ is given by the expression:

$$\chi(q) = \sum_j \langle \sigma_0 \sigma_j \rangle \, e^{2\pi i q z_j} \tag{4}$$

where the sum is over all correlations $\langle \sigma_0 \sigma_j \rangle$ and z_j is the number of lattice spacings in the z direction, between the spins 0 and j. The correlation function itself is given by:

$$\langle \sigma_0 \sigma_j \rangle = \mathrm{Tr}(e^{-\beta \mathcal{H}} \sigma_0 \sigma_j) / \mathrm{Tr}(e^{-\beta \mathcal{H}}) \tag{5}$$

and it is well known (Domb 1974) that a high temperature expansion for this quantity can be constructed in which the various terms are associated with graphs on the lattice. A variant of this method, which is particularly convenient for Ising problems with a variety of interaction types, was derived by the present author (Oitmaa 1981). This results in an expansion of the form:

$$\chi(q) = 1 + 2 \sum_{\{G\}} W_G X_G(q, \{v_\alpha\}) \tag{6}$$

where the sum is over a particular set of graphs (of which there are 3296 through 11th order), W_G is a 'graph weight' which is the same for any Ising problem, and X_G is a factor which is obtained by:

(i) summing over all embeddings of G on the lattice,

(ii) including for each embedding a factor $\Pi_\alpha v_\alpha^{n_\alpha}$, where $v_\alpha = \tanh \beta J_\alpha$ and n_α is the number of type α bonds in the embedding,

(iii) including for each embedding a factor $e^{2\pi i q z}$.

In this way we obtain a high temperature series of the form:

$$\chi(q) = 1 + 2 \sum_{n=1}^{\infty} \sum_{\substack{r,s \\ (r+s \leq n)}} C_{nrs}(q) v_0^{n-r-s} v_1^r v_2^s \tag{7}$$

with $v_0 = \tanh \beta J_0$, etc. The data is actually stored in the form $C_{nrs}(z)$ for correlations between spins which are separated by any number of z units. The data is much too extensive to publish here, but can be supplied on request. It is then possible to construct a χ series for any wavenumber.

There are several partial checks available for our results. For the ANNNI model itself Redner and Stanley (1977a, b) have tabulated the C_{nrs} coefficients for the ferromagnetic case ($q = 0$) for $n \leq 8$, as well as coefficients for the second and fourth moment $\langle z^2 \rangle$, $\langle z^4 \rangle$. Our results are in complete agreement to this order (we have found two minor typographical errors in the $\langle z^4 \rangle$ coefficients). For the additional terms there is a check with published series only for the case $J_2 = 0$, and in this case complete agreement is found. There must remain the possibility that our results contain small errors, not detectable by the above checks, but we feel this is unlikely as most steps in the series calculation are computerised.

For purposes of the analysis we parametrise the interactions by $J_1 = \kappa J_0$, $J_2 = \lambda J_0$. For any choice (κ, λ) we expand the hyperbolic tangent factors and obtain a single variable series in powers of $K = J_0/k_B T$. These are then analysed by standard methods.

3. The three-dimensional case

We discuss in this section the analysis of our series for the three-dimensional ANNNI model, and present the form of the resulting phase diagram. For simplicity we discuss only the case $J_1 = J_0$ (i.e. $\kappa = 1$) although series for other values of κ have been analysed, with similar results.

For $\lambda > 0$ (all interactions ferromagnetic) the series for $\chi(0)$ show a consistent physical singularity whose position and exponent can be easily and reliably estimated from conventional ratio and Padé approximate methods. The ratios do show an odd–even oscillation, reflecting the presence of an 'antiferromagnetic' singularity on the negative real axis, which approaches the circle of convergence as $\lambda \to 0$. For $-0.26 \leqslant \lambda < 0$ the analysis is complicated by the fact that the antiferromagnetic singularity lies closer to the origin than the singularity of interest. An Euler transformation can be used to move the physical singularity closer to the origin, as discussed for example by Redner and Stanley (1977a). The transition is still to a ferromagnetic ordered phase.

When the interaction J_2 becomes sufficiently negative ($\lambda \leqslant -0.27$) the ordered phase is no longer the uniform ferromagnetic one ($q = 0$) but one characterised by a modulation of wavenumber q. This is clearly manifested by the ferromagnetic series becoming highly irregular and yielding no consistent singularity on the positive real axis. Within a rather small range of non-zero q values the series for $\chi(q)$ is found to be regular and can be analysed by first carrying out an Euler transformation and then using both ratio and Padé methods. The critical wavenumber q_c can be estimated from the position of the minimum in a plot of $K_c(q)$ against q. A much simpler criterion for locating the approximate value of q_c is to simply plot the values of the high order series coefficients against q. The coefficients go through a distinct maximum which, if the function contained only a single power-law singularity on the positive real axis, would give q_c precisely.

We present some typical results, for the case $\lambda = -0.3$. Figure 2(a) shows the variation of series coefficients with q, and shows a clear maximum at $q \simeq 0.08$, which tends to move slightly to lower q at higher order. This suggests that $\chi(q)$ for $q \simeq 0.08$ will show a divergence with the smallest value of K_c, which can indeed be seen in the Padé results of figure 2(b). Although the minima for different Padés do not occur at the same value of q, when we consider in addition Padés and ratios to the Euler transformed series we are able to estimate, for this case, that $q_c = 0.07 \pm 0.005$. A similar analysis has been carried out for values of λ in the range $\lambda \geqslant -0.8$.

Our results are summarised in figure 3. Figure 3(a) shows the variation of the critical temperature with the parameter $\lambda = J_2/J_0$. The shape of the critical line is quite similar to the mean-field prediction and the location of the Lifshitz point, which we estimate to be $\lambda = -0.270 \pm 0.005$, is close to the mean-field value $\lambda = -0.25$. Figure 3(b) gives the variation of critical wavenumber q_c with λ, and shows that q_c increases smoothly from zero at $\lambda = \lambda_L$ to a value $q_c \simeq 0.2$ at $\lambda = -0.8$. This shows that the transition is not immediately to the $\langle 2 \rangle$ phase which has $q_c = 0.25$, and which is the stable phase at $T = 0$ for $\lambda < -0.5$, but rather to a modulated phase with a longer wavelength. Using this approach we cannot of course distinguish between a truly incommensurate 'floating' phase and a sequence of commensurate phases of varying order.

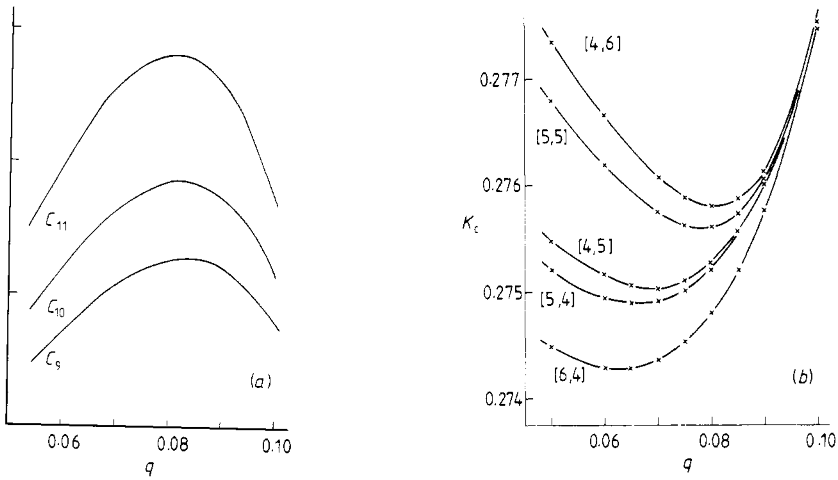


Figure 2. (*a*) Values of series coefficients C_9, C_{10}, C_{11} (on an arbitrary scale) against q for the 3D system with $\lambda = -0.3$. (*b*) Padé estimates of K_c against q for the 3D system with $\lambda = -0.3$.

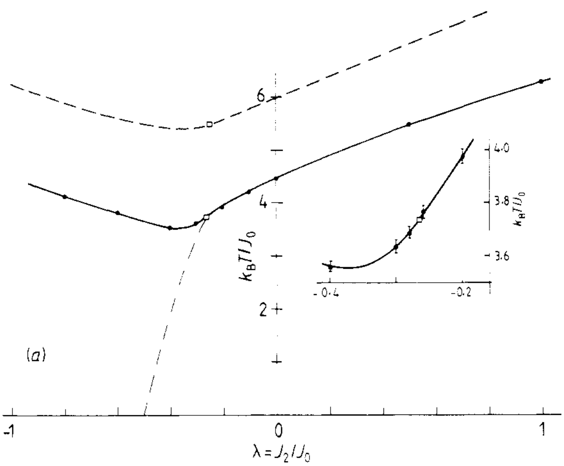
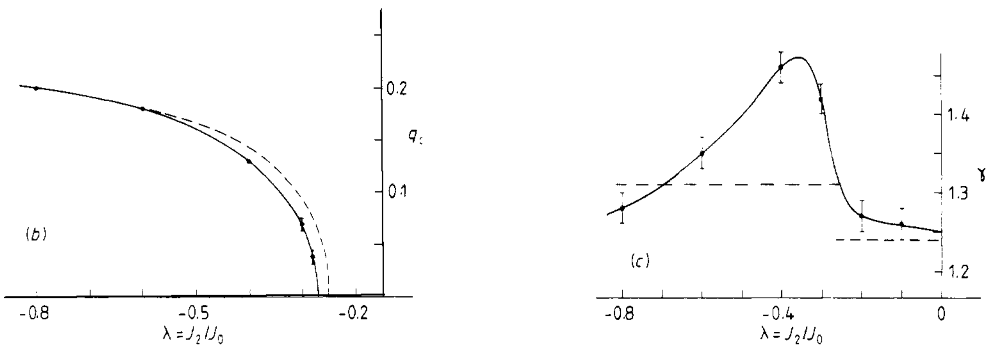


Figure 3. (*a*) Variation of critical temperature with J_2/J_0 for the 3D ANNNI model, as determined from high temperature series. The open square denotes the Lifshitz point and the broken curve gives the mean-field results. (*b*) Variation of critical wavenumber with J_2/J_0. The broken curve is the mean-field result. (*c*) Apparent variation of exponent γ with J_2/J_0. The broken line shows the behaviour expected from universality and renormalisation group arguments.



We expect that the nature of the singularity in the susceptibility, along the entire critical line, should be of the form

$$\chi(q) = C\{1 - [K/K_c(q)]\}^{-\gamma}.$$

The exponent γ should remain constant at the Ising value $\gamma \cong 1.24$ for $\lambda > \lambda_L$, should take a different value γ_L at the Lifshitz point itself, and should have a constant value $\gamma \cong 1.32$ for $\lambda < \lambda_L$, corresponding to the XY exponent (Garel and Pfeuty 1976). Analysis of the series by standard ratio and Padé methods, assuming a simple power-law singularity, yields an apparent continuous variation of γ, as shown in figure 3(c). Our results do not differ significantly from those of Redner and Stanley (1977b), who attributed the apparent continuous variation in γ to crossover effects and to the shortness of the series. A puzzling and unexplained feature is the apparent decrease of γ to something like the Ising value for large negative λ.

It is conceivable that the estimates of γ are being affected by the presence of a confluent singularity. In order to test this we have used a quadratic transformation of the form

$$1 - (K/K_c) = p^2(1-y)^2/(p-y)^2$$

as proposed by Nickel and Dixon (1982). The transformed series are not significantly more regular than the series in K, and although estimates of γ tend to be slightly below those of figure 3(c), the change is about the same for all λ.

4. The two-dimensional case

Analysis of the series for the two-dimensional ANNNI model follows similar lines, but the results are less precise. We discuss again only the case in which $J_1 = J_0$.

For $\lambda > -0.2$ analysis of the ferromagnetic series $\chi(0)$ is straightforward and confirms the picture of a normal Ising-like transition, with $\gamma = 1.75$, and a critical temperature which varies with λ in the manner shown in figure 5. Below $\lambda = -0.2$ the series become irregular and cannot be analysed directly. We have attempted to analyse the series for

$$\hat{\chi}(K') = (1-2K')^{-1.75}\chi(K') \qquad (9)$$

with $K' = K/(1+2K)$. The Euler transformation moves the interfering antiferromagnetic singularity far outside the circle of convergence, and the prefactor removes the spurious singularity introduced by the Euler transformation (Redner and Stanley 1977a).

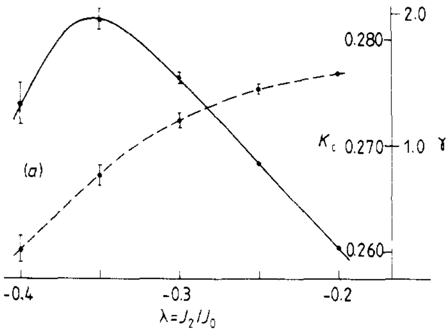
Padé approximants to $\hat{\chi}(K')$ reveal a very consistent pole on the positive real axis, but the results are unusual for two reasons:

(i) the location of the pole moves to larger values as λ is decreased from -0.2 to -0.35 (indicating a decreasing critical temperature, as expected) but begins to move back towards the origin for $\lambda < -0.35$,

(ii) the exponent decreases rapidly with λ.

These results are shown in figure 4(a), and could be interpreted as indicating the presence of a finite temperature Lifshitz point at $\lambda \simeq 0.3$. However, ratio plots for the same series, shown in figure 4(b), do not appear to converge towards the Padé predictions, at least for $\lambda < -0.3$. This suggests that the series in this region are too short to reveal the true asymptotic behaviour. If there were a Lifshitz point at $\lambda < -0.3$

then it should be possible to extend the critical line past this point by looking for singularities in a $\chi(q)$ with non-zero q. However, we find that series for all q are highly irregular, and we thus conclude that there is no evidence for a Lifshitz point in this region.



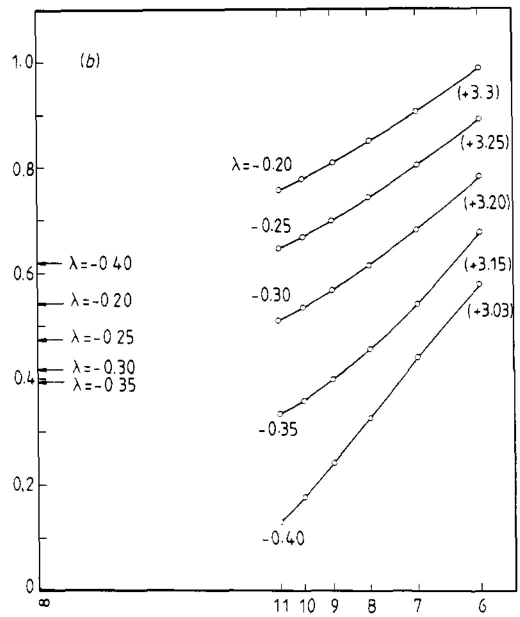


Figure 4. (a) Position of pole on positive real axis for $-0.4 \leq \lambda \leq -0.2$ for the 2D ANNNI model (full curve) and the apparent susceptibility exponent (broken curve). (b) Ratio plot of the transformed series $\hat{\chi}(K)$ for $-0.4 \leq \lambda \leq -0.2$. The ordinate scale has been adjusted by subtracting the numbers in brackets from the true ratios. The arrows on the ordinate axes show the Padé estimates.

For large negative values of λ ($\lambda \leq -0.6$) we find evidence for a consistent singularity in $\chi(q)$ with a non-zero wavenumber q, indicating that the transition in this case is to a modulated phase. For a given λ the critical wavenumber q_c is determined in the same way as for the three-dimensional case in § 3. We give some details of the analysis for the case $\lambda = -1.0$, which is typical. In this case q_c is found to be $\simeq 0.23$. Dlog Padés to $\chi(K)$ give a consistent pole at $K_c \simeq 0.485 \pm 0.01$ with an exponent $\gamma \simeq 1.9$, and an antiferromagnetic singularity at $K \simeq -0.28$. After an Euler transformation $K' = K/(1+3K)$ Dlog Padés to $\chi(K')$ give a very consistent pole at $K_c' = 0.1980 \pm 0.0001$ (corresponding to $K_c = 0.4877 \pm 0.0006$) and an exponent $\gamma = 1.94 \pm 0.01$. Ratio analysis gives results consistent with this.

There are strong arguments for expecting the transition to the modulated phase to be of the Kosterlitz–Thouless type, with an essential singularity rather than one of the usual power-law type. Guttmann (1978) has pointed out that one might distinguish between the two cases by analysing the second logarithmic derivative, when an algebraic singularity in the original series would become a simple pole with residue 1 whereas an essential singularity of the form $\exp\{c[1-(x/x_c)]^{-g}\}$ would give a pole with residue $1+g$. In the present case $(\mathrm{Dlog})^2$ Padés to $\chi(K)$ and to the transformed series $\chi(K')$ give poles near the previous values (but less consistent) and exponents in the range 1.00 ± 0.02. The series evidence thus favours a conventional power-law singularity.

Series for the range $-1.5 \leq \lambda \leq -0.6$ have been analysed in this way and the estimated critical temperatures are shown in figure 5, together with the values of the critical wavenumber q_c. The values of q_c are in very good agreement with those obtained by other methods (Selke 1981, Duxbury and Barber 1982). There is an apparent small

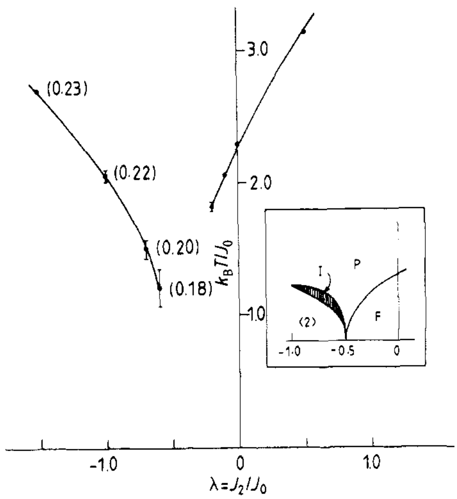


Figure 5. Variation of critical temperature with J_2/J_0 for the 2D ANNNI model, as determined from a high temperature series. The numbers in brackets are the values of the critical wavenumber q_c. The inset shows the generally accepted form of the 2D phase diagram, showing ferromagnetic (F), incommensurate (I), and commensurate ⟨2⟩ phases.

variation in the exponent γ along the critical line, with $\gamma \simeq 1.75$ at $\lambda = -1.5$ increasing to $\gamma \simeq 2.15$ at $\lambda = -0.6$. There is no unusual behaviour in the region $\lambda \simeq -1.1$ which might indicate the existence of a Lifshitz point marking the junction of incommensurate, paramagnetic, and modulated ⟨2⟩ phases. Such a Lifshitz point has been predicted in some previous work. For $\lambda > -0.6$ the series become irregular for all wavenumbers and it has not been possible to carry out any convincing analysis.

5. Conclusions

We have presented results of a study of the ANNNI model in both three and two dimensions, based on eleven-term high temperature series for the wavevector dependent susceptibility. The results extend previous series studies and complement those obtained by other methods. The main advantage of series methods is that one can, in many cases, obtain very accurate estimates of critical temperatures and exponents. However, the method does have limitations in situations where complex critical behaviour is expected, as in the vicinity of multicritical points, since the available series are too short to reveal the true asymptotic behaviour in such cases.

For the three-dimensional case our results are, we feel, reliable over the entire range of interaction parameters, and provide accurate estimates of the boundaries of the ferromagnetic and modulated phases, as well as the location of the Lifshitz point. It must be admitted, with hindsight, that the addition of three additional terms to the series of Redner and Stanley (1977b) has produced no really significant change in their results. The rather unusual behaviour of the exponent γ along the paramagnetic/modulated line still remains.

For the two-dimensional case we have given the only published series results, although Redner has obtained ten-term series several years ago. In this case there is a range of parameters $-0.6 \leqslant \lambda \leqslant -0.3$ for which the series are too irregular to allow definite conclusions to be drawn. We interpret our results of figure 5 to be consistent with the currently accepted picture in which the paramagnetic phase extends to zero temperature at the multiphase point $\lambda = -0.5$. Our results favour a power-law singularity along the paramagnetic/incommensurate line with an exponent γ in the range 1.75-2.15, rather than an essential singularity of the Kosterlitz-Thouless type. There is no evidence for a finite temperature Lifshitz point on the incommensurate side of the phase diagram.

Our results clearly demonstrate that the critical behaviour of this model in two and three dimensions is quite different. Further analysis of the series may be able to resolve some of the puzzles which still remain. A number of series are given in the appendix and other attempts to analyse them would be welcomed. Series for other cases can be obtained upon request. Studies using different techniques, such as Monte Carlo renormalisation (Swendsen 1979), or a special purpose computer (Hoogland *et al* 1983) may also answer some of the unresolved questions.

Acknowledgments

I would like to thank a number of colleagues, in particular R J Elliott, S Redner and W Selke, for helpful discussions.

Appendix

Table 1. Susceptibility series for the three-dimensional model.

$\lambda = 1, q = 0$	$\lambda = -0.2, q = 0$	$\lambda = -0.25, q = 0$
1.000 000 000 00E0	1.000 000 000 00E0	1.000 000 000 00E0
8.000 000 000 00E0	5.600 000 000 00E0	5.500 000 000 00E0
5.600 000 000 00E1	2.528 000 000 00E1	2.412 500 000 00E1
3.833 333 333 33E2	1.127 093 333 33E2	1.044 791 666 67E2
2.574 666 666 67E3	4.781 290 666 67E2	4.277 526 041 67E2
1.715 306 666 67E4	2.039 174 314 67E3	1.765 612 239 58E3
1.133 844 888 89E5	8.444 735 621 69E3	7.027 978 146 70E3
7.459 306 349 21E5	3.528 699 075 87E4	2.837 678 530 35E4
4.886 583 377 78E6	1.446 100 073 01E5	1.115 588 367 37E5
3.191 478 445 01E7	5.983 568 056 28E5	4.459 481 893 69E5
2.078 971 450 36E8	2.435 438 857 98E6	1.737 880 894 12E6
1.351 506 363 43E9	1.001 946 221 65E7	6.912 021 026 13E6
$\lambda = -0.3, q = 0.07$	$\lambda = -0.4, q = 0.13$	$\lambda = -0.8, q = 0.20$
1.000 000 000 00E0	1.000 000 000 00E0	1.000 000 000 00E0
5.427 199 711 08E0	5.419 326 627 48E0	5.912 461 179 75E0
2.327 449 670 40E1	2.304 910 109 53E1	2.767 719 720 20E1
9.842 540 304 59E1	9.664 521 728 53E1	1.290 286 148 93E2
3.908 949 392 04E2	3.784 272 153 43E2	5.655 526 537 41E2
1.568 406 293 77E3	1.502 517 521 69E3	2.534 053 661 88E3
6.019 952 475 72E2	5.659 067 980 61E3	1.081 410 306 86E4
2.357 132 601 81E4	2.193 871 046 76E4	4.779 306 252 94E4
8.898 546 940 33E4	8.095 533 450 93E4	2.008 901 594 30E5
3.447 113 753 57E5	3.111 257 174 66E5	8.845 306 830 19E5
1.284 292 586 25E6	1.127 015 525 33E6	3.663 052 538 83E6
4.955 670 586 96E6	4.337 064 913 75E6	1.619 764 452 80E7

Table 2. Susceptibility series for the two-dimensional model.

$\lambda = 1, q = 0$	$\lambda = -0.2, q = 0$	$\lambda = -0.3, q = 0$
1.000 000 000 00E0	1.000 000 000 00E0	1.000 000 000 00E0
6.000 000 000 00E0	3.600 000 000 00E0	3.400 000 000 00E0
3.000 000 000 00E1	8.880 000 000 00E0	7.380 000 000 00E0
1.420 000 000 00E2	2.113 600 000 00E1	1.531 066 666 67E1
6.460 000 000 00E2	4.232 640 000 00E1	2.345 340 000 00E1
2.860 800 000 00E3	8.955 084 800 00E1	4.264 228 533 33E1
1.239 133 333 33E4	1.761 849 941 33E2	6.365 392 080 00E1
5.282 660 952 38E4	3.616 261 234 59E2	1.206 617 475 63E2
2.225 032 952 38E5	6.987 397 650 05E2	1.760 274 274 80E2
9.283 588 613 76E5	1.398 914 420 46E3	3.455 550 842 37E2
3.843 439 686 77E6	2.657 642 389 95E3	4.500 903 505 23E2
1.580 957 573 26E7	5.275 689 383 03E3	1.002 642 939 97E3
$\lambda = -0.4, q = 0$	$\lambda = -0.6, q = 0.18$	$\lambda = -1.0, q = 0.22$
1.000 000 000 00E0	1.000 000 000 00E0	1.000 000 000 00E0
3.200 000 000 00E0	3.616 467 370 83E0	4.234 315 600 95E0
5.920 000 000 00E0	8.358 836 244 27E0	1.192 942 860 84E1
1.010 133 333 33E1	1.884 434 970 97E1	3.392 994 912 87E1
8.219 733 333 33E0	3.111 313 678 94E1	7.823 423 083 54E1
1.156 420 266 67E1	6.581 053 046 06E1	2.126 912 352 83E2
8.298 609 778 05E−1	9.165 608 431 28E1	4.397 595 748 62E2
2.041 867 524 07E1	2.012 541 588 12E2	1.197 761 272 18E3
−7.577 164 621 13E0	2.042 922 085 67E2	2.245 785 495 75E3
8.480 364 098 15E1	6.017 247 298 68E2	6.516 892 569 93E3
−8.477 105 136 84E1	2.423 560 593 08E2	1.044 532 497 78E4
3.716 329 053 94E2	2.089 270 682 22E3	3.530 374 358 55E4

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